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ON THE CONSISTENCY AND EQUIVALENCE OF CERTAIN GENERALIZED DEFINITIONS OF THE LIMIT OF A FUNCTION OF A CONTINUOUS VARIABLE.

BY L. L. SILVERMAN.

§ 1. **Introduction.**[*] In a recent paper Hurwitz and Silverman[†] have studied certain definitions of summability of sequences, and have obtained criteria for the consistency and for the equivalence of these definitions. Two definitions are *consistent* if whenever each of the definitions gives a value to a sequence, the two values are the same; two consistent definitions are *equivalent* when each evaluates every sequence evaluated by the other. In the present paper[‡] corresponding criteria are obtained for the case of certain generalized definitions of a limit of a function of a continuous variable.

In a previous article[§] the author has studied the transformation[||]

$$v(x) = \alpha u(x) + \int_0^x k(x, y)u(y)dy, \tag{1}$$

where $u(x)$ is bounded and integrable, $0 \leq x \leq x_1$, $k(x, y)$ is integrable in y for each x, $0 < \delta \leqq y \leqq x$, and $\int_0^x |k(x, y)|dy$ converges[¶] for each x. The function $k(x, y)$ is the *kernel* and the number α the *coefficient* of the transformation. We shall denote by the symbol $[\alpha, k(x, y)]$ the transformation whose coefficient is α and whose kernel is $k(x, y)$. The transformation is regular if the existence of

$$\lim_{x=\infty} u(x)$$

implies the existence of

$$\lim_{x=\infty} v(x)$$

and the equality of the limits. Examples of regular transformations are the identity

$$(E) \qquad \alpha = 1, \qquad k(x, y) = 0$$

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[*] Presented to the American Mathematical Society, December, 1915.

[†] Transactions of the American Mathematical Society, vol. 18 (1917), p. 1.

[‡] The author is greatly indebted to Professor W. A. Hurwitz for many valuable suggestions in connection with this paper.

[§] Transactions of the American Mathematical Society, vol. 17 (1916), p. 284.

[||] The lower limit of integration might be any number, but is taken zero for convenience.

[¶] The convergence of the integral is assumed in order to give a meaning to formula 1.

and the transformation

$$(M) \qquad \alpha = 0, \qquad k(x, y) = \frac{1}{x}, \qquad 0 < x.$$

The transformations E and M correspond in the case of sequences to convergence and to summability of the first order, respectively. To obtain the analog of Cesàro-summability of higher orders, we define with Landau*

$$v_0(x) = u(x), \qquad v_n(x) = \frac{n!}{x^n} S_n(x), \qquad x > 0,$$

where

$$S_0(x) = u(x), \qquad S_n(x) = \int_0^x S_{n-h}(y)dy; \qquad n = 1, 2, 3, \cdots.$$

Given $u(x)$, we can by this definition find $v_n(x)$. To obtain the functional relation between $v_n(x)$ and $u(x)$ in the form (1), we shall prove the formula

$$(2) \qquad v_n(x) = \frac{n!}{x^n} \int_0^x \frac{(x-y)^{h-1}}{(h-1)!} S_{n-h}(y)dy, \qquad h = 1, 2, 3, \cdots, n.$$

Let n be any fixed integer, and assume that formula (2) holds for some fixed h; then expressing S_{n-h} in terms of S_{n-h-1}, we have

$$\begin{aligned} v_n(x) &= \frac{n!}{x^n} \int_0^x \int_0^y \frac{(x-y)^{h-1}}{(h-1)!} S_{n-h-1}(z)dzdy \\ &= \frac{n!}{x^n} \int_0^x \int_z^x \frac{(x-y)^{h-1}}{(h-1)!} S_{n-h-1}(z)dydz \\ &= \frac{n!}{x^n} \int_0^x \frac{(x-z)^h}{h!} S_{n-h-1}(z)dz, \end{aligned}$$

which is formula (2) when h is replaced by $h + 1$. Since the formula obviously holds for $h = 1$, it is true in general. Letting $h = n$ in (2), we obtain

$$v_n(x) = \frac{n!}{x^n} \int_0^x \frac{(x-y)^{n-1}}{(n-1)!} u(y)dy,$$

which is of the form (1). We have thus found

$$(C_n) \quad \alpha = 0, \qquad c_n(x, y) = \frac{n(x-y)^{n-1}}{x^n}, \qquad 0 < x, \qquad 0 \leqq y \leqq x,$$

the coefficient and the kernel of the transformation C_n, corresponding to the case of Cesàro summability of order n.

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* Sächsische Berichte, vol. 65 (1913), p. 131.

The analog to Hölder summability is defined* by the formulæ

$$v_0(x) = u(x), \qquad v_n(x) = \frac{1}{x}\int_0^x v_{n-1}(y)dy, \qquad n = 1, 2, 3, \cdots.$$

Since $u(x)$ is bounded, $0 \leqq x \leqq x_1$, it follows that $v_n(x)$ is bounded and continuous, $0 \leqq x \leqq x_1$, $n = 1, 2, \cdots$. To obtain the functional relation between $v_n(x)$ and $u(x)$ in the form (1), we shall prove the formula

$$(3) \quad v_n(x) = \frac{1}{(h-1)!\,x}\int_0^x \log^{h-1}\left(\frac{x}{y}\right) v_{n-h}(y)dy, \qquad h = 1, 2, 3, \cdots, n.$$

We shall first show that the integral in (3) converges. If M is an upper bound of $u(x)$, we have

$$\frac{1}{(h-1)!\,x}\int_0^x \log^{h-1}\left(\frac{x}{y}\right) |v_{n-h}(y)|dy \leqq \lim_{\delta \doteq 0}\frac{M}{(h-1)!\,x}\int_\delta^x \log^{h-1}\left(\frac{x}{y}\right)dy$$

$$= \lim_{\delta \doteq 0}\left[M\frac{y}{x}\sum_{p=0}^{h-1}\frac{\log^p\left(\frac{x}{y}\right)}{p!}\right]_{y=\delta}^{y=x} = M[1-0] = M,$$

so that the integral in (3) converges. Now let n be any fixed number, and assume that formula (3) holds for some h; then expressing v_{n-h} in terms of v_{n-h-1}, we have

$$v_n(x) = \lim_{\delta \doteq 0}\frac{1}{(h-1)!\,x}\int_\delta^x\int_0^y \frac{1}{y}\log^{h-1}\left(\frac{x}{y}\right)v_{n-h-1}(z)dzdy$$

$$= \lim_{\delta \doteq 0}\frac{1}{(h-1)!\,x}\int_\delta^x\int_z^x \frac{1}{y}\log^{h-1}\left(\frac{x}{y}\right)v_{n-h-1}(z)dydz$$

$$+ \lim_{\delta \doteq 0}\frac{1}{(h-1)!\,x}\int_\delta^x\int_0^\delta \frac{1}{y}\log^{h-1}\left(\frac{x}{y}\right)v_{n-h-1}(z)dzdy.$$

Since

$$\lim_{\delta = 0}\left|\frac{1}{(h-1)!\,x}\int_\delta^x\int_0^\delta \frac{1}{y}\log^{h-1}\left(\frac{x}{y}\right)v_{n-h-1}(z)dzdy\right|$$

$$\leqq \lim_{\delta \doteq 0}\frac{M}{(h-1)!\,x}\int_0^\delta\int_\delta^x \frac{1}{y}\log^{h-1}\left(\frac{x}{y}\right)dydz = \lim_{\delta \doteq 0}\frac{M\delta}{h!\,x}\log^h\left(\frac{x}{\delta}\right) = 0,$$

it follows that

$$v_n(x) = \frac{1}{(h-1)!\,x}\int_0^x v_{n-h-1}(z)\int_z^x \frac{1}{y}\log^{h-1}\left(\frac{x}{y}\right)dydz$$

$$= \frac{1}{h!\,x}\int_0^x \log^h\left(\frac{x}{z}\right)v_{n-h-1}(z)dz,$$

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* Landau, loc. cit.

which is formula (3) when h is replaced by $h + 1$. Since the formula holds for $h = 1$, it is true in general. Letting $h = n$ in (3), we obtain

$$v_n(x) = \frac{1}{(n-1)!\,x}\int_0^x \log^{n-1}\left(\frac{x}{y}\right) u(y)dy, \qquad 0 < y \leqq x,$$

which is of the form (1). We have thus found

$$(H_n) \quad \alpha = 0, \qquad h_n(x, y) = \frac{1}{(n-1)!\,x}\log^{n-1}\left(\frac{x}{y}\right), \qquad 0 < y \leqq x,$$

the coefficient and the kernel of the transformation H_n, corresponding to the case of Hölder summability of order n.

Let $k(x, y)$ be integrable in y for each x, $0 < y \leqq x$; then *a sufficient condition that* $[\alpha, k(x, y)]$ *correspond to a regular transformation** *is*

$$(a) \qquad |k(x, y)| \leqq \frac{N}{x^{1-\rho}y^{\rho}},$$

$$(b) \qquad \lim_{x=\infty}\int_0^x k(x, y)dy = 1 - \alpha,$$

where $0 \leqq \rho < 1$ *and* $N \geqq 0$.

It is easily verified that the transformations C_n and H_n satisfy these conditions for regularity. We shall be concerned in this paper with only those kernels which satisfy conditions (a) and (b).

When the integral equation (1) possesses a solution in the form

$$(2) \qquad u(x) = \beta v(x) + \int_0^x l(x, y)v(y)dy,$$

we shall call $l(x, y)$ the *inverse kernel*,† and the transformation (2) the inverse *transformation*; symbolically, if $v(x) = A[u(x)]$, then

$$u(x) = A^{-1}[v(x)].$$

If $v(x) = A[u(x)]$ and $w(x) = B[v(x)]$, then $w(x) = B[A(u(x)]$, the transformation being BA. If A and B correspond to $[\alpha, k(x, y)]$ and $[\beta, l(x, y)]$ respectively, $aA + bB$ will correspond to $[a\alpha + b\beta, ak(x, y) + bl(x, y)]$. If $A_1, A_2, \cdots$ correspond to $[\alpha_1, k_1(x, y)], [\alpha_2, k_2(x, y)], \cdots$, respectively, $a_1A_1 + a_2A_2 + \cdots$ will correspond to $[\alpha, k(x, y)]$, whenever

$$\lim_{n=\infty}[a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n] = \alpha,$$

$$\lim_{n=\infty}[a_1k_1(x, y) + a_2k_2(x, y) + \cdots + a_nk_n(x, y)] = k(x, y),$$

the last limit existing uniformly‡ in y for each x.

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* Bulletin of the American Mathematical Society, vol. 22, p. 461. For other forms of necessary and sufficient conditions see article in Transactions, vol. 17.

† In the theory of integral equations the kernel in (1) is usually defined to be $-(1/\alpha)\,k(x, y)$, and the inverse kernel, usually called the resolvent function, is then $\alpha l(x, y)$.

‡ The uniform approach of this limit is assumed in order to ensure the integrability of $k(x, y)$.

§ 2. **A special class of transformations.** It will be seen that if A and B are regular, then for any constant a, AB and $aA + (1 - a)B$ are regular. Landau* has studied the regular transformation

$$aE + (1 - a)M.$$

It is natural to consider the more general regular transformation

$$a_0E + a_1M + a_2M^2 + \cdots + a_nM^n, \qquad a_0 + a_1 + a_2 + \cdots + a_n = 1,$$

or, more generally, the symbol

$$a_0E + a_1M + a_2M^2 + \cdots + a_nM^n + \cdots,$$

and to ask under what conditions the symbol defines a regular transformation.

THEOREM 1. *If $f(z) = a_0 + a_1z + a_2z^2 + \cdots$ is analytic within and on the boundary of the unit circle, and if $f(1) = 1$, then the symbol $a_0E + a_1M + a_2M^2 + \cdots$ defines a regular transformation.*

To prove this theorem we shall show that the sufficient conditions for regularity (a) and (b) are satisfied. Starting with the inequality†

$$\frac{1}{n!}\log^n(t) \leqq \frac{1}{\rho^n}t^\rho, \qquad t \geqq 1, \qquad 0 < \rho < 1, \qquad n = 1, 2, 3, \cdots,$$

letting $t = x/y$, and dividing by x, we have

$$h_{n+1}(x, y) = \frac{1}{n!\,x}\log^n\left(\frac{x}{y}\right) \leqq \frac{1}{\rho^n}\frac{1}{x^{1-\rho}y^\rho}, \qquad 0 < y \leqq x, \qquad 0 < \rho < 1;$$

so that

$$|k(x, y)| \leqq \Sigma\ |a_n| h_n(x, y) \leqq \frac{\rho}{x^{1-\rho}y^\rho}\sum_{n=1}^{\infty}\frac{|a_n|}{\rho^n},$$

where the series

$$\sum_{n=1}^{\infty}\frac{|a_n|}{\rho^n}$$

surely converges for values of ρ sufficiently near unity, since $\sum_{n=0}^{\infty} a_nz^n$ is analytic on the circumference of the unit circle. Letting the value of the series $\sum_{n=1}^{\infty} |a_n|/\rho^n$, for a fixed admissible ρ, equal to m, we have

$$|k(x, y)| \leqq \frac{m\rho}{x^{1-\rho}y^\rho}$$

which shows that condition (a) is satisfied.

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* Sächsische Berichte, loc. cit.

† This inequality is easily proved by mathematical induction. Assuming it to hold for $n = k$, dividing by t, and integrating from 1 to t, we see that it holds for $n = k + 1$. To show that it holds for $n = 1$, we observe that for $t = 1$, we get $0 \leqq 1/\rho$, and for $t > 1$, the derivative of the left is less than the derivative of the right side of the inequality.

To prove that condition (*b*) is satisfied, we observe that, since

$$h_{n+1}(x, y) \leqq \frac{1}{\rho^n} \frac{1}{x^{1-\rho} y^\rho}, \qquad 0 < \delta \leqq y \leqq x, \qquad 0 < \rho < 1,$$

$$\leqq \frac{1}{\rho^n} \frac{1}{x^{1-\rho} \delta^\rho}$$

the series

$$\sum_{n=1}^{\infty} a_n h_n(x, y) = k(x, y)$$

converges uniformly for each x, $\delta \leqq y \leqq x$. Hence we may integrate term by term, and

$$\int_\delta^x k(x, y) dy = \sum_{i=1}^{\infty} a_i \int_\delta^x h_i(x, y) dy,$$

and the series of integrals converges uniformly in δ. We accordingly have*

$$\lim_{\delta \doteq 0} \int_\delta^x k(x, y) dy = \sum_{i=1}^{\infty} a_i \lim_{\delta \doteq 0} \int_\delta^x h_i(x, y) dy$$

$$= \sum_{i=1}^{\infty} a_i = f(1) - a_0 = 1 - a_0.$$

COROLLARY 1. *The kernel of the transformation M^n is given by the formula*

$$(4) \qquad h_n(x, y) = \frac{1}{2\pi i x} \int_C t^{n-2} \left(\frac{x}{y}\right)^{1/t} dt, \qquad 0 < x, \qquad 0 < y \leqq x,$$

where C is any circle including the origin.

For

$$\frac{1}{2\pi i x} \int_C t^{n-2} \left(\frac{x}{y}\right)^{1/t} dt = \frac{1}{2\pi i x} \int_C t^{n-2} \sum_{p=0}^{\infty} \frac{1}{p!} \frac{1}{t^p} \log^p \left(\frac{x}{y}\right) dt$$

$$= \frac{1}{2\pi i x} \sum_{p=0}^{\infty} \frac{1}{p!} \log^p \left(\frac{x}{y}\right) \int_C t^{n-2-p} dt,$$

since the first series cnoverges uniformly in t, its general term being less in absolute value than $N^p \log^p (x/y)/p!$, where N is the maximum of $|1/t|$ on C. Since

$$\int_C t^{n-2-p} dt = 0, \qquad p \neq n - 1,$$

$$= 2\pi i, \qquad p = n - 1,$$

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* By a well-known theorem (see Osgood, Funktionentheorie, vol. 1, 2d ed., p. 593), the order of two limits, to be performed in succession, may be interchanged, provided each limit exists and one of them exists uniformly.

it follows that

$$\frac{1}{2\pi i x}\int_C t^{n-2}\left(\frac{x}{y}\right)^{1/t} dt = \frac{1}{x}\frac{1}{(n-1)!}\log^{n-1}\left(\frac{x}{y}\right) = h_n(x, y).$$

COROLLARY 2. *The transformation* $[\alpha, k(x, y)]$ *is given in terms of* $f(z)$, *a function analytic in a circle* C *including the origin, by the formulae,*

$$(5)\quad \alpha = f(0), \qquad k(x, y) = \frac{1}{2\pi i x}\int_{C_1}\frac{f(t)}{t^2}\left(\frac{x}{y}\right)^{1/t} dt, \qquad 0 < x, \qquad 0 < y \leqq x,$$

where C_1 *is a circle including the origin entirely inside* C.

Obviously $\alpha = a_0 = f(0)$; it remains to derive the expression for $k(x, y)$. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then by the preceding corollary*

$$\begin{aligned} k(x, y) &= \sum_{n=1}^{\infty} a_n h_n(x, y) = \frac{1}{2\pi i x}\sum_{n=0}^{\infty}\int_{C_1} a_n t^{n-2}\left(\frac{x}{y}\right)^{1/t} dt \\ &= \frac{1}{2\pi i x}\int_{C_1}\frac{1}{t^2}\sum_{n=1}^{\infty} a_n t^n\left(\frac{x}{y}\right)^{1/t} dt \\ &= \frac{1}{2\pi i x}\int_{C_1}\frac{1}{t^2}[f(t) - a_0]\left(\frac{x}{y}\right)^{1/t} dt \\ &= \frac{1}{2\pi i x}\int_{C_1}\frac{f(t)}{t^2}\left(\frac{x}{y}\right)^{1/t} dt. \end{aligned}$$

Definition. The transformation A *corresponds* to $f(z)$, a function analytic in some circle C including the origin, if the coefficient α and the kernel $k(x, y)$ of the transformation are given by (5).

COROLLARY 3. *The kernel of the transformation corresponding to*

$$g(z) = \frac{n!\, z^n}{(1 + z)(1 + 2z)\cdots(1 + \overline{n-1}\, z)}$$

is

$$C_n(x, y) = \frac{n}{x}\left(1 - \frac{y}{x}\right)^{n-1}, \qquad 0 < x, \qquad 0 \leqq y \leqq x.$$

For let $k(x, y)$ be the kernel corresponding to $g(z)$; then by the preceding corollary

$$k(x, y) = \frac{n!}{2\pi i x}\int_{C_1}\frac{t^{n-2}}{(1 + t)(1 + 2t)\cdots(1 + \overline{n-1}\, t)}\left(\frac{x}{y}\right)^{1/t} dt$$

provided C_1 is a circle about the origin of radius less than $1/(n - 1)$.

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* Term by term integration is justified by the uniform convergence of the series whose general term $a_n t^n (x/y)^{1/t}$ is less in absolute value than $|a_n| b^n (x/y)^c$, where b, the maximum of $|t|$ on C_1, lies inside C, and $c \geqq R(1/t)$ on C_1.

Since*

$$\frac{t^{n-2}}{(1+t)(1+2t)\cdots(1+\overline{n-1}\,t)}$$

$$=\sum_{j=1}^{n-1}(-1)^{j-1}\frac{1}{(j-1)!(n-j-1)!}\frac{1}{(jt+1)},$$

hence

$$k(x,y)=\frac{n!}{2\pi i x}\int_{C_1}\sum_{j=1}^{n-1}(-1)^{j-1}\frac{1}{(j-1)!(n-j-1)!}\frac{1}{jt+1}\left(\frac{x}{y}\right)^{1/t}dt$$

$$=\frac{n!}{2\pi i x}\sum_{j=1}^{n-1}(-1)^{j-1}\frac{1}{(j-1)!(n-j-1)!}\int_{C_1}\frac{1}{jt+1}\left(\frac{x}{y}\right)^{1/t}dt.$$

Letting $t=1/s$ and C' the inverse of C_1, we have

$$\int_{C_1}\frac{1}{jt+1}\left(\frac{x}{y}\right)^{1/t}dt=\int_{C'}\frac{1}{(j+s)s}\left(\frac{x}{y}\right)^{s}ds=\frac{2\pi i}{j}\left(1-\frac{y^j}{x^j}\right).$$

Hence

$$k(x,y)=\frac{n!}{2\pi i x}\sum_{j=1}^{n-1}(-1)^{j-1}\frac{1}{(j-1)!(n-j-1)!}\frac{2\pi i}{j}\left(1-\frac{y^j}{x^j}\right)$$

$$=\frac{n}{x}\sum_{j=1}^{n-1}(-1)^{j-1}\frac{(n-1)!}{j!(n-j-1)!}\left[1-\left(\frac{y}{x}\right)^j\right]$$

$$=\frac{n}{x}\sum_{j=0}^{n-1}(-1)^{j}\frac{(n-1)!}{j!(n-j-1)!}\left(\frac{y}{x}\right)^j$$

$$=\frac{n}{x}\left(1-\frac{y}{x}\right)^{n-1}=C_n(x,y).$$

THEOREM 2. *If A and B correspond to $f(z)$ and $g(z)$ respectively, then BA corresponds to $g(z)f(z)$.*

Let C correspond to $g(z)f(z)$; then we are to prove that $C=BA$. If $k(x,y)$, $l(x,y)$, $m(x,y)$ are the kernels of A, B, C respectively, then the transformations A and BA are given as follows:

$$v(x)=f(0)u(x)+\int_0^x k(x,y)u(y)dy,$$

$$w(x)=g(0)v(x)+\int_0^x l(x,y)v(y)dy$$

$$=g(0)f(0)u(x)+\int_0^x[g(0)k(x,y)+f(0)l(x,y)]u(y)dy$$

$$+\int_0^x\int_0^y l(x,y)k(y,s)u(s)dsdy.$$

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* This formula is easily obtained by resolving into partial fractions.

Interchanging the order of integration* in the double integral, and letting

$$(6) \qquad q(x, s) = \int_s^x l(x, y)k(y, s)dy,$$

we obtain

$$w(x) = g(0)f(0)u(x) + \int_0^x [g(0)k(x, y) + f(0)l(x, y) + q(x, y)]u(y)dy.$$

Since the coefficient of the transformation BA is seen to be $g(0)f(0)$, which is the same as that of the transformation C, it remains only to prove the kernels of C and BA are identical, i.e.,

$$(7) \qquad m(x, y) = g(0)k(x, y) + f(0)l(x, y) + q(x, y).$$

From (5) and (6) we obtain

$$\begin{aligned} -4\pi^2 xq(x, y) &= -4\pi^2 x \int_y^x l(x, s)k(s, y)ds \\ &= \int_y^x \left[\int_{C_1} \frac{g(t_1)}{t_1^2}\left(\frac{x}{s}\right)^{1/t_1} dt_1\right] \frac{1}{s} \int_{C_2} \frac{f(t_2)}{t_2^2}\left(\frac{s}{y}\right)^{1/t_2} dt_2 ds \\ &= \int_{C_1} \frac{g(t_1)}{t_1^2} x^{1/t_1} \int_{C_2} \frac{f(t_2)}{t_2^2} y^{-1/t_2} \int_y^x s^{1/t_2 - 1/t_1 - 1} ds dt_2 dt_1 \\ &= \int_{C_1}\int_{C_2} \frac{g(t_1)f(t_2)}{t_1 t_2(t_1 - t_2)}\left[\left(\frac{x}{y}\right)^{1/t_2} - \left(\frac{x}{y}\right)^{1/t_1}\right] dt_2 dt_1, \end{aligned}$$

where C_1 and C_2 are circles including the origin. Taking C_1 inside C_2, we have

$$\begin{aligned} \int_{C_1}\int_{C_2} \frac{g(t_1)f(t_2)}{t_1 t_2(t_1 - t_2)}\left(\frac{x}{y}\right)^{1/t_2} dt_2 dt_1 &= \int_{C_2} \frac{f(t_2)}{t_2}\left(\frac{x}{y}\right)^{1/t_2}\left[\int_{C_1} \frac{g(t_1)}{t_1(t_1 - t_2)} dt_1\right] dt_2 \\ &= -2\pi i g(0) \int_{C_2} \frac{f(t_2)}{t_2^2}\left(\frac{x}{y}\right)^{1/t_2} dt_2 \end{aligned}$$

and

$$\begin{aligned} \int_{C_1}\int_{C_2} \frac{g(t_1)f(t_2)}{t_1 t_2(t_1 - t_2)}\left(\frac{x}{y}\right)^{1/t_1} &= -\int_{C_1} \frac{g(t_1)}{t_1}\left(\frac{x}{y}\right)^{1/t_1}\left[\int_{C_2} \frac{f(t_2)}{t_2(t_2 - t_1)} dt_2\right] dt_1 \\ &= -2\pi i \int_{C_1} \frac{g(t_1)}{t_1}\left(\frac{x}{y}\right)^{1/t_1}\left[\frac{f(t_1) - f(0)}{t_1}\right] dt_1. \end{aligned}$$

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* The change in the order of integration is easily justified if we bear in mind that the kernels $k(x, y)$, $l(x, y)$ are subject to the conditions of p. 131.

Hence

$$- 4\pi^2 x q(x, y) = - 2\pi i \int_C \frac{g(0)f(t) + f(0)g(t)}{t^2} \left(\frac{x}{y}\right)^{1/t} dt$$

$$+ 2\pi i \int_C \frac{g(t)f(t)}{t^2} \left(\frac{x}{y}\right)^{1/t} dt,$$

or

$$- 4\pi^2 x q(x, y) = 4\pi^2 x[g(o)k(x, y) + f(0)l(x, y)] - 4\pi^2 x m(x, y)$$

which reduces to (7).

Corollary. *If A and B correspond to $f(z)$ and $g(z)$ respectively, each of the functions being analytic in some circle including the origin, then $AB = BA$.*

§ 3. **A more general class of transformations.** We have seen in the preceding section that if the function $f(z)$ is analytic in the unit circle, then the corresponding transformation is regular. We shall now consider more generally the function $f(z)$ analytic in a circle C, of radius $\frac{1}{2}$ about the point $\frac{1}{2}$, and define the corresponding transformation $f(M)$ by (5), where the circle C_1 is now to be taken of radius $\frac{1}{2} + \epsilon$ ($\epsilon > 0$) about the point $\frac{1}{2}$.

Theorem 3. *If $f(z)$ is analytic inside and on the boundary of the circle of radius $\frac{1}{2}$ about the point $\frac{1}{2}$, and if $f(1) = 1$, then the transformation corresponding to $f(z)$ is regular.*

We shall show that the sufficient conditions for the regularity of a transformation, referred to in the first section, are satisfied. By (5) we have

$$\int_\delta^x k(x, y)dy = \frac{1}{2\pi i x} \int_\delta^x \int_{C_1} \frac{f(t)}{t^2} \left(\frac{x}{y}\right)^{1/t} dt dy$$

$$= \frac{1}{2\pi i x} \int_{C_1} \frac{f(t)}{t^2} \int_\delta^x \left(\frac{x}{y}\right)^{1/t} dy dt$$

$$= \frac{1}{2\pi i} \int_{C_1} \frac{f(t)}{t^2\left(1 - \frac{1}{t}\right)} \left[1 - \left(\frac{\delta}{x}\right)^{1-1/t}\right] dt.$$

Now $|f(t)/t^2(1 - 1/t)| = |f(t)/t(t - 1)|$ is continuous along C_1 and accordingly has a maximum N. Hence

$$\left|\int_{C_1} \frac{f(t)}{t(t - 1)} \left(\frac{\delta}{x}\right)^{1-1/t} dt\right| \leqq N \int_{C_1} \left|\left(\frac{\delta}{x}\right)^{1-1/t}\right| |dt|$$

$$= N \int_{C_1} \left(\frac{\delta}{x}\right)^{R(1-1/t)} |dt| \leqq 2\pi N \left(\frac{\delta}{x}\right)^{1+1/\epsilon} (\tfrac{1}{2} + \epsilon);$$

for, since t lies on the circumference of C_1,

$$\frac{\epsilon}{1+\epsilon} \leqq R\left(1 - \frac{1}{t}\right) \leqq 1 + \frac{1}{\epsilon}.$$

We have also

$$\int_{C_1} \frac{f(t)}{t(t-1)}\, dt = 2\pi i[f(1) - f(0)].$$

Hence

$$\int_0^x k(x, y)dy = \lim_{\delta=0} \int_\delta^x k(x, y)dy$$

$$= f(1) - f(0) = 1 - f(0) = 1 - \alpha,$$

since by definition $\alpha = f(0)$. Thus condition (b) is satisfied.

Secondly, letting N_1 represent the maximum of $f(t)/t^2$ on C_1, we have

$$|k(x, y)| \leqq \frac{1}{2\pi x} \int_{C_1} \left|\frac{f(t)}{t^2}\right| \left|\frac{x}{y}\right|^{1/t} |dt| \leqq \frac{N_1}{2\pi x} \int_{C_1} \left(\frac{x}{y}\right)^{R(1/t)} |dt|.$$

Since

$$R\left(\frac{1}{t}\right) \leqq \frac{1}{1+\epsilon},$$

we obtain

$$|k(x, y)| \leqq \frac{N_1}{2\pi x} \int_{C_1} \left(\frac{x}{y}\right)^{1/1+\epsilon} |dt|$$

$$= \frac{N_1(\frac{1}{2} + \epsilon)}{x^{\epsilon/1+\epsilon} y^{1/1+\epsilon}}.$$

Hence, letting

$$N_1(\tfrac{1}{2} + \epsilon) = N, \qquad \frac{1}{1+\epsilon} = \rho,$$

$$|k(x, y)| \leqq \frac{N}{x^{1-\rho} y^\rho}, \qquad 0 < \rho < 1,$$

which is condition (a). The transformation has accordingly been shown to be regular.

We shall now consider the function $f(z)$ analytic in C except for poles, and study the corresponding symbol $f(M)$.

Lemma 1. The function $f(z) = (1 - \rho)/(z - \rho)$, where ρ is a point inside or on the boundary of the circle C, does not define a regular transformation.

The proposition is obviously true for the cases $\rho = 0$ and $\rho = 1$. In every other case the function $f(z)$ defines a transformation $f(M)$, which we shall show is not regular. To show this, define

$$v(x) = x^p, \qquad p = \frac{1}{\rho} - 1,$$

and find $u(x)$ by applying to $v(x)$ the transformation corresponding to

$$\frac{1}{f(z)} = \frac{z - \rho}{1 - \rho}.$$

Thus

$$\begin{aligned} u(x) &= \frac{-\rho}{1-\rho} v(x) + \frac{1}{(1-\rho)x}\int_0^x v(y)dy \\ &= \frac{-\rho}{1-\rho} x^p + \frac{1}{(1-\rho)x}\int_0^x y^p dy \\ &= \left(-\rho + \frac{1}{p+1}\right)\frac{x^p}{1-\rho} = 0. \end{aligned}$$

Hence

$$\lim_{x=\infty} u(x) = 0.$$

On the other hand, from the restrictions on ρ,

$$R(p) \geqq 0;$$

accordingly, since

$$\begin{aligned} \lim_{x=\infty} |x^p| &= \infty, \qquad R(p) > 0 \\ |x^p| &= 1, \qquad R(p) = 0, \end{aligned}$$

it follows that

$$\lim_{x=\infty} v(x)$$

is not zero. Hence the function $(1 - \rho)/(z - \rho)$ does not define a regular transformation.

THEOREM 4. *If $f(z)$ has at least one pole in C, but is analytic except for poles within and on the boundary of C, and if $f(1) = 1$, then $f(z)$ does not define a regular transformation.*

The proof of this theorem, based on function-theoretical considerations, is identical with the one given for the case of sequences in the paper referred to above,* and is accordingly omitted here.

§ 4. **Consistency and Equivalence of Regular Transformation.** We shall now use the term *analytically regular* to describe a transformation $f(M)$ corresponding to a function $f(z)$, analytic throughout C, and such that $f(1) = 1$. A number of properties of such transformations follows immediately from the results of the preceding sections.

THEOREM 5. *All analytically regular transformations are consistent.*

Let A and B, two analytically regular transformations, evaluate $\lim_{x=\infty} u(x)$ to u_1 and u_2 respectively; then BA and AB evaluate this limit to u_1 and u_2 respectively. By the corollary to Theorem 2, $AB = BA$; hence $u_1 = u_2$.

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* Transactions (1917), p. 13.

THEOREM 6. *If $f(M)$ and $g(M)$ are analytically regular transformations, a necessary and sufficient condition that $f(M)$ should evaluate every expression* $\lim_{x=\infty} u(x)$ *which $g(M)$ evaluates, giving it the same value, is that all the zeros of $g(z)$ in C, should be zeros of at least as high order of $f(z)$.*

The proof of this theorem and that of the next theorem are omitted, being the same as those in the case of sequences in the paper already mentioned.

THEOREM 7. *If $f(M)$ and $g(M)$ are analytically regular, a necessary and sufficient condition that they be equivalent is that $f(z)$ and $g(z)$ have in C the same zeros with the same orders.*

COROLLARY 1. *A necessary and sufficient condition that the analytically regular transformation $f(M)$ be reversible (equivalent to the identical transformation) is that $f(z)$ does not vanish in C.*

COROLLARY 2. *The Hölder and Cesàro transformations of like order are equivalent.*

From corollaries 1 and 3 of Theorem 1, the transformations H_n, C_n correspond respectively to the functions

$$z^n, \frac{n!\,z^n}{(1+z)(1+2z)\cdots(1+\overline{n-1}\,z)}.$$

Each of these functions is analytic in C, having no zero except $z = 0$, which is in both cases a zero of order n. Hence the two transformations are equivalent.